Quantum-group description of decaying particles

1998 J. Phys. A: Math. Gen. 31 9117

(http://iopscience.iop.org/0305-4470/31/46/003)

J. Phys. A: Math. Gen. **31** (1998) 9117–9122. Printed in the UK PII: S0305-4470(98)94008-2

Quantum-group description of decaying particles

M Arik, U Kayserilioğlu and G Ünel

Received 8 May 1998, in final form 18 August 1998

Abstract. The particle decay problem in the random-set approach leads naturally to the q-Poisson distribution. Motivated by the fact that in the classical quantum mechanical description of particle decay one ends up with a composite system, we construct a Hilbert space using the eigenstates of the average particle number operator N. In this approach, time is represented by integer multiples of an observation quantum τ. It is shown that these considerations lead to the redefined q-oscillator and an element of $SU_q(n)$ acts like a reversal operator on n time-ordered states.

Consideration of random sets in the definition of new discrete probabilities has hinted at the applicability of these probability distributions to interesting problems in physics. It has been shown that in the context of random sets the non-extensivity of classical set theory gives rise to a unitary quantum-group symmetry associated with the fact that when random sets are joined the total number of elements is independent of the order in which this is carried out [1]. On the other hand, q-deformed Stirling numbers, which are related to q-bosons [2, 3], turn out to be also important from the point of view of random sets [4].

In this paper we wish to apply the random-set approach to the particle decay problem. We start by defining a finite source set S of M elements, a target set A and a probability $\mathrm{d}p = \lambda\,\mathrm{d}t$, the probability of making a choice from the set S in an infinitesimal time interval $\mathrm{d}t$. At time $t = 0$, the set A will be null. The chosen elements of S will only be added to A if and only if they already do not exist in A. Note that this is in accordance with the mathematical definition of a set. The probability of having m elements in the set A at time t will be denoted by $P_m(t)$. This probability can be defined by using a simple iteration: In order to have $m+1$ elements in the set A at time $t+\mathrm{d}t$, one has either m elements at time t and makes a choice from S which does not already exist in A during the interval $\mathrm{d}t$, or one has already $m+1$ elements in A at time t. For the latter case, one either should not make a choice or should choose an element which already exists in the set A. This definition gives the following differential recurrence relation for $P_m(t)$:

$$P_{m+1}(t+\mathrm{d}t) = \lambda\,\mathrm{d}t\left(1-\frac{m}{M}\right)P_m(t) + \left[(1-\lambda\,\mathrm{d}t) + \lambda\,\mathrm{d}t\,\frac{m+1}{M}\right]P_{m+1}(t). \tag{1}$$

If the source set S is taken to represent M radioactive nuclei, assuming that each nucleus can only decay once, $P_m(t)$ becomes the probability of having m decayed nuclei at time t. It is possible to solve (1) to obtain an explicit formula for $P_m(t)$:

$$P_m(t) = \binom{M}{m}\left(\mathrm{e}^{-\lambda t/M}\right)^{M-m}\left(1-\mathrm{e}^{-\lambda t/M}\right)^m \tag{2}$$

0305-4470/98/469117+06$19.50

which can also be obtained by the following reasoning. The decay process is a Poisson process and initially the decay width for the whole set is λ. Since there are M elements in the set S the decay width for one particle is λ/M. Hence the probability that this particle will decay in time interval t is $p = 1 - e^{-\lambda t/M}$. Thus $P_m(t)$ is a binomial distribution for M trials, each trial being the decay of one of the M particles. In order to end up with m decayed nuclei, we have to choose m particles out of M with $M - m$ of them undecayed and m decayed. This yields equation (2) exactly. Note that the usual Poisson distribution is obtained as M goes to infinity. Hence we will call $P_m(t)$ the *q-Poisson distribution*. Here we should emphasize that this distribution is different to that appearing in previous literature [5].

The average number of decayed particles is given by

$$\langle m \rangle = \sum_{m=0}^{M} m P_m(t) = M\left(1 - e^{-\lambda t/M}\right) = -\frac{1 - q^{\lambda t}}{\ln q} \tag{3}$$

with $q = e^{-1/M}$. Note that at time $t = 0$ all particles are undecayed and the set A is empty.

Motivated by the analogous problem of particle decay in quantum mechanics, we now would like to set up a Hilbert space in which we can represent our decaying nuclei. At the time $t = 0$, the state associated with the decay process, and thus with the target set A, will be represented by the ground state, $|0\rangle$, since in our initial configuration we have started with undecayed particles. As time passes, the nuclei will decay one by one and finally as time $t \to \infty$, the target set A will have the same cardinality as the source set S. Thus, it is feasible to denote the final state at time $t = \infty$ by $|M\rangle$. It may be worth noting that the creation operator acting on the state vector is also associated in some way with the flow of time, since we are continuously monitoring the decaying nuclei in time. The states between $t = 0$ and $t = \infty$ can be expressed by the use of a density matrix according to the classical quantum mechanical approach:

$$\rho(t) = \sum_{m=0}^{M} |m\rangle\ P_m(t)\ \langle m|. \tag{4}$$

If we take the dynamical variable of this system we are observing as the number of decayed particles, we should associate a Hermitian operator with it in order to be able to measure it:

$$N|m\rangle = m|m\rangle \qquad m = 0, 1, 2, \ldots, M. \tag{5}$$

The average number of decayed particles then becomes $\langle m \rangle = \mathrm{Tr}(N\rho(t))$. However, in this standard quantum mechanical description, our system is a composite one and a particle, while decaying, cannot be represented by a pure state which is an eigenstate of N.

In physics, to simplify a problem, it is best to work in the simplest and most convenient basis. Starting from this idea, we would like to prepare a recipe in which even a single particle, while decaying, can be represented by a pure state. Since a state is, in fact, identified with the probability distributions of its observables, we argue that we should build a new Hilbert space which will be composed of the eigenstates of an operator corresponding to the average number of decayed particles. In quantum mechanics a Hermitian operator is associated with each observable. The result of this new approach is to associate a Hermitian operator with the average number of decayed particles which is the only relevant, physically measurable quantity in this problem.

To be able to measure the average number of decayed particles one needs, in fact, a small amount of time in which the averaging process is performed. We will denote this quantum of time by τ and represent our conventional time by multiples of it:

$$|t\rangle = |n\tau\rangle \qquad n = 0, 1, 2, \ldots\ . \tag{6}$$

Since our goal was to represent even a single decaying particle by a pure state, we will reduce the problem of M decaying nuclei to the decay problem of a single nucleus. This single nucleus, evolving in time, will be the system under observation. The action of the operator N, measuring the average number of decayed particles on this system will be the $M \to 1$ limit of equation (3), and instead of using a particular notation to represent the state of our decaying nucleus we will simply write states in which time is the only parameter. This notation is convenient for this problem, since the evolution of the nucleus under study depends only on time. The operator N which describes the average number of decayed particles satisfies

$$N|n\tau\rangle = \left(1 - \mathrm{e}^{-\lambda n\tau}\right)|n\tau\rangle. \tag{7}$$

The above equation is valid for any n and defines a parameter q given by $q = \mathrm{e}^{-\lambda\tau}$. These time states are eigenstates of a Hermitian operator with different eigenvalues; thus they are orthonormal. The creation operator $a^\dagger$ which usually causes the state under investigation to evolve by one step, can now be thought of as the time evolution operator, since we have chosen to represent the state of our nucleus only by the evolution of time. It will cause an increment in time by one observation quantum τ:

$$a^\dagger|n\tau\rangle = \alpha_n|(n+1)\tau\rangle \tag{8}$$

where α_n is arbitrary. One can take the Hermitian conjugate of the above equation to find the action of a, the Hermitian conjugate of $a^\dagger$ as

$$a|n\tau\rangle = \overline{\alpha}_{n-1}|(n-1)\tau\rangle. \tag{9}$$

We see that the annihilation operator a alters the current time state or equivalently the state of the nucleus under investigation by one observation unit τ in the backward direction. The resemblance of this duality to the Feynman interpretation of antiparticles hints at the possibility of developing a formalism where $a^\dagger$ will be related to the time evolution of particles, whereas a will be related to the time evolution of antiparticles. Naturally, the average number of decayed particles is related to time, and as usual we define N as

$$N = a^\dagger a. \tag{10}$$

The arbitrary factor α arising in the definition of creation/annihilation operators can now be found by the use of equations (7), (8), (9), (10):

$$\begin{aligned} &N|n\tau\rangle = a^\dagger a|n\tau\rangle = |\alpha_{n-1}|^2\,|n\tau\rangle \\ &|\alpha_{n-1}| = \sqrt{1-q^n}. \end{aligned} \tag{11}$$

The phase of these coefficients, α_n, can be absorbed into the vectors $|n\tau\rangle$ without changing their orthonormality. Equation (11) enables us to find the commutation relation between a and $a^\dagger$:

$$(aa^\dagger - qa^\dagger a)|n\tau\rangle = (1-q)|n\tau\rangle. \tag{12}$$

This is indeed the redefined q-oscillator which is well known in the literature [6]. In the commutation relation implied by (12)

$$aa^\dagger - qa^\dagger a = 1 - q \tag{13}$$

a can be represented by the q-difference operator, defined on functions $f(z)$ by

$$D\,f(z) = z^{-1}(f(z) - f(qz)) \tag{14}$$

whereas $a^\dagger$ can be represented by the multiply-by-z operator, defined by

$$Z\,f(z) = zf(z) \tag{15}$$

which satisfies

$$(DZ - qZD)\, f(z) = (1-q) f(z). \tag{16}$$

For $z \in \mathbb{C}$, a coherent-state representation can be constructed [6].

One might expect from the beginning a non-commuting algebra for the time evolution operators by arguing on the physical principle of causality. However, it is interesting to see that this simple example yielded the q-oscillator algebra as the associated quantum mechanical algebraic framework.

Up to this point we have concentrated on the average number of the decayed particles. It is also possible to deal with the number of undecayed particles. Since we are now working with a single particle, this number is given by $1 - N$. The eigenvalues of this new operator, which we prefer to denote as $b^\dagger b$, will clearly be q^n. The fact that we are dealing with a single particle can now be written algebraically as

$$a^\dagger a + b^\dagger b = 1. \tag{17}$$

The action of the operators b and $b^\dagger$ on the particle states, or equivalently on the time states, cannot be that of a lowering or raising operator, simply because $b^\dagger b$ does not have a zero eigenvalue over the states $|n\tau\rangle$ for any finite n. However, it does have a zero eigenvalue on the state $|\infty\rangle$ which corresponds to $n \to \infty$. Acting on this state by b gives $b|\infty\rangle = 0$, so this state can be thought of as the ground state of the operator b. However, $b^\dagger$ cannot be a lowering or raising operator on this state since $\infty \pm 1 = \infty$. Thus, the simplest action which can be associated with these operators is a change in the normalization of a state. Representing the phase of this factor by $\mathrm{i}\beta$ we can define the action of the operator b on the eigenstate as

$$b|n\tau\rangle = q^{n/2} \mathrm{e}^{\mathrm{i}\beta} |n\tau\rangle. \tag{18}$$

Since we know the action of the operators a, b and their Hermitian conjugates on the states which form our Hilbert space, we can now proceed to calculate the commutation relations between these operators. In fact, they satisfy the very well known quantum-group algebra with parameter $\tilde{q} = \sqrt{q}$:

$$\begin{aligned} ab &= \tilde{q}\, ba \\ ab^\dagger &= \tilde{q}\, b^\dagger a \\ bb^\dagger &= b^\dagger b \\ aa^\dagger + \tilde{q}^2\, bb^\dagger &= a^\dagger a + b^\dagger b = 1 \end{aligned} \tag{19}$$

so that the 2×2 matrix constructed from these four operators will be an element of the quantum group $SU_{\tilde{q}}(2)$:

$$U_2 = \begin{pmatrix} a & -\tilde{q}\, b \\ b^\dagger & a^\dagger \end{pmatrix} \in SU_{\tilde{q}}(2). \tag{20}$$

Note that the state $|\infty\rangle$ mentioned in the above paragraph is a one-dimensional representation of this algebra that satisfies

$$a|\infty\rangle = a^\dagger|\infty\rangle = |\infty\rangle \tag{21}$$

$$b|\infty\rangle = b^\dagger|\infty\rangle = 0. \tag{22}$$

The 2×2 matrix U_2 naturally acts on two time states that we will bring together as a column matrix and name a doublet. At the beginning we will form our doublet from arbitrary states to have a general insight on the action of the operator U_2:

$$\begin{pmatrix} a & -\tilde{q}\,b \\ b^\dagger & a^\dagger \end{pmatrix}\begin{pmatrix} |m\tau\rangle \\ |n\tau\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{1-q^m}|(m-1)\tau\rangle - \sqrt{q^{n+1}}\,\mathrm{e}^{\mathrm{i}\beta}|n\tau\rangle \\ \sqrt{q^m}\,\mathrm{e}^{-\mathrm{i}\beta}|m\tau\rangle + \sqrt{1-q^{n+1}}|(n+1)\tau\rangle \end{pmatrix}. \tag{23}$$

For ordered time states, i.e. for $n = m-1$, the above complicated formula boils down to a much simpler one:

$$U_2\begin{pmatrix} |m\tau\rangle \\ |(m-1)\tau\rangle \end{pmatrix} = \begin{pmatrix} c_1|(m-1)\tau\rangle \\ c_0|m\tau\rangle \end{pmatrix} \tag{24}$$

with

$$c_0 = \sqrt{1-q^m} - \sqrt{q^m}\mathrm{e}^{\mathrm{i}\beta}$$

$$c_1 = \sqrt{q^m}\mathrm{e}^{-\mathrm{i}\beta} + \sqrt{1-q^m}.$$

One can observe that except for the coefficients c_1 and c_0, the action of the operator U_2 is to reverse the ordering of these states. Thus one can think the operator U_2 as a projective 'time reversal operator' since it returned us, up to some coefficients, the time-ordered particle states in reverse order. In practice, by choosing an appropriate value for the arbitrary phase factor in equation (14), one can simplify these coefficients. For example, for $\beta = \pi/2$, one obtains

$$c = c_0 = c_1 = \sqrt{1-q^m} - \mathrm{i}\sqrt{q^m}. \tag{25}$$

Thus we can write the action of the operator U_2 on time-ordered states as

$$U_2\begin{pmatrix} |m\tau\rangle \\ |(m-1)\tau\rangle \end{pmatrix} = c\begin{pmatrix} |(m-1)\tau\rangle \\ |m\tau\rangle \end{pmatrix}. \tag{26}$$

The role of $SU_q(2)$, i.e. the gathering of operators a, $a^\dagger$, b, $b^\dagger$ into an $SU_q(2)$ matrix as in (20), is not essential for the particle decay problem. However, it is interesting that this arrangement acts as a time reversal operator. This behaviour of interchanging the order of the states in a doublet deserves more attention and we will investigate it further. However, from now on we will only be interested in the projective properties of the operator U_2 and of the other operators which will be derived from it. To study the action of such an operator on state triplets, we consider the matrices

$$A_1 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{U_2} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right) \qquad A_2 = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{U_2}} \\ 0 & & \end{array}\right) \tag{27}$$

then $A = A_1 \dot{\otimes} A_2 \dot{\otimes} A_1$ becomes an element of $SU_q(3)$. Here '$\otimes$' denotes the tensor product and '$\cdot$' denotes the matrix product as usual. On the other hand, the operator $U_3 = A_1 \cdot A_2 \cdot A_1$ acts on the time-ordered triplets as an order reversing operator up to some coefficients which we omit:

$$U_3\begin{pmatrix} |m\tau\rangle \\ |(m-1)\tau\rangle \\ |(m-2)\tau\rangle \end{pmatrix} \sim \begin{pmatrix} |(m-2)\tau\rangle \\ |(m-1)\tau\rangle \\ |m\tau\rangle \end{pmatrix}. \tag{28}$$

Similarly, one can construct a generalized operator U_n using the building-block matrices of $SU_q(n)$ [7] and taking their matrix products. This generalized matrix operator U_n will act on ordered state n-tuples. We would like to emphasize that this U_n is not an element of the

quantum group $SU_q(n)$. For example, $SU_q(3)$ is obtained by matrix multiplying $SU_q(2)$ matrices of the form (27) where different matrices have independent oscillator structure that commute. However to obtain (28) the matrices A_1 and A_2 should be taken to belong to the same oscillator algebra (19) in the matrices U_2 given by (20). The action of U_n can be written projectively (up to complex factors multiplying the states) as

$$U_n \begin{pmatrix} |m\tau\rangle \\ \vdots \\ |(m-n+1)\tau\rangle \end{pmatrix} \sim \begin{pmatrix} |(m-n+1)\tau\rangle \\ \vdots \\ |m\tau\rangle \end{pmatrix}. \tag{29}$$

Since we have seen that this generalized operator reverses the time-ordered states of our system, we will call it a projective time reversal operator. Admittedly, this concept of a time reversal operator is rather unconventional. However, it comes out naturally from the further investigation of the solution of our initial problem: representing a single decaying nucleus by the eigenstates of a Hermitian operator. While studying this problem, we have encountered the concept of the time observation quantum τ and the quantum-group structure. A requirement of this quantum-group description of decaying particle is to fix the time that the observation starts and to name that instant $t = 0$, so that our particle decays slowly starting from $t = 0$ up to $t = \infty$, instead of a classical, abrupt, unpredictable decay. We believe that this new description is interesting and is open to further development.

References

[1] Arik M, Kornfilt J and Yildiz A 1998 *Phys. Lett.* A **235** 318
[2] Katriel J and Duchamp G 1995 *J. Phys. A: Math. Gen.* **28** 7209
[3] Katriel J and Kibler M 1992 *J. Phys. A: Math. Gen.* **25** 2683
[4] Arik M and Unel G 1998 *J. Phys. A: Math. Gen.* **31** 1121
[5] Adam G and Katriel J 1974 *Lett. Nuovo Cimento* **13** 565
[6] Vaksman L L and Soibelman Y S 1988 *Funkt. Anal. Priloz.* **22** 1
[7] Arik M and Celik S 1993 *Z. Phys.* C **59** 99